Q1

i) $\boxed{2\cos 2x - 7e^{7x}}$ CR

ii) $u = x^2$ $\quad v = \ln x$ PR

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{x^2}{x} + (\ln x)2x = x + 2x\ln x = x(1 + 2\ln x)$$

$$\frac{dy}{dx} = \boxed{x(1 + 2\ln x)}$$

CHAIN RULE

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

PRODUCT RULE

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

iii) $u = \cos 3x$ $\quad v = \tan 2x$ QR, CR

$\frac{du}{dx} = 3(-\sin 3x)$ $\quad \frac{dv}{dx} = 2(\sec^2 2x)$

$$\frac{dy}{dx} = \frac{\tan 2x(-3\sin 3x) - \cos 3x(2\sec^2 2x)}{\tan^2 2x}$$

$$\frac{dy}{dx} = \boxed{\frac{-3\sin 3x\tan 2x - 2\sec^2 2x\cos 3x}{\tan^2 2x}}$$

iv) $\frac{dy}{dx} = \boxed{\frac{\sec^2 x}{\tan x}}$ CR

Q2

$$y - y_1 = m(x - x_1)$$

when $x = 1$

$$y = e^{-3(1)} + \ln 1 = e^{-3}$$

$$m = \frac{dy}{dx} = -3e^{-3x} + \frac{1}{x}$$

when $x = 1$

$$m = -3e^{-3(1)} + \frac{1}{(1)} = -3e^{-3} + 1$$

$$y - e^{-3} = (-3e^{-3} + 1)(x - 1)$$

$$y = -3e^{-3}x + 3e^{-3} + x - 1 + e^{-3}$$

$$= (-3e^{-3} + 1)x + (4e^{-3} - 1)$$

$$= \left(\frac{-3}{e^3} + \frac{e^3}{e^3}\right)x + \left(\frac{4}{e^3} - \frac{e^3}{e^3}\right)$$

$$\boxed{y = \left(\frac{-3 + e^3}{e^3}\right)x + \left(\frac{4 - e^3}{e^3}\right)}$$

Q3

Method 1: Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let $y = \ln u$ $\quad u = ax^n$

$\frac{dy}{du} = \frac{1}{u}$ $\quad \frac{du}{dx} = anx^{n-1}$

$$\frac{dy}{dx} = \frac{1}{u} anx^{n-1}$$

$$= \frac{\cancel{a}nx^{n-1}}{\cancel{a}x^n} = nx^{-1} = \frac{n}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{n}{x}}$$

OR

Method 2: Laws of Logs

$$y = \ln(ax^n) = \ln a + \ln x^n = \ln a + n\ln x$$

$$\boxed{\frac{dy}{dx} = \frac{n}{x}}$$

Q4

Find the gradient of the normal to the curve $y = 5\cos(e^x - \frac{\pi}{2})$ at the point with $x$-coordinate 0. Give your answer correct to 3 decimal places.

[4]

$$m_{normal} = \frac{-1}{m_{tangent}} = \frac{-1}{\frac{dy}{dx}}$$

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let $y = 5\cos u$ $\quad u = e^x - \frac{\pi}{2}$

$\frac{dy}{du} = -5\sin u$ $\quad \frac{du}{dx} = e^x$

$$\frac{dy}{dx} = -5(\sin u)e^x$$

$$= -5(\sin(e^x - \tfrac{\pi}{2}))e^x$$

Sub in $x = 0$

$$\frac{dy}{dx} = -5(\sin(e^0 - \tfrac{\pi}{2}))e^0$$

$$= -5\sin(1 - \tfrac{\pi}{2})$$

$$= 2.70\ldots$$

$$m_{normal} = \frac{-1}{2.70\ldots} = \boxed{-0.370} \text{ (3 dp)}$$

Q5a

a)

Product rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let $u = 2\sin 3x - \cos 3x \qquad v = e^{6-x}$

$\frac{du}{dx} = 6\cos 3x + 3\sin 3x \qquad \frac{dv}{dx} = -e^{6-x}$

$$\frac{d(uv)}{dx} = (2\sin 3x - \cos 3x)(-e^{6-x}) + (e^{6-x})(6\cos 3x + 3\sin 3x)$$

$$= e^{6-x}(-2\sin 3x + \cos 3x + 6\cos 3x + 3\sin 3x)$$

$$= \boxed{e^{6-x}(\sin 3x + 7\cos 3x)}$$

Q5b

b)

Product rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Let $u = (x^2 - x)^2 \qquad v = \ln 5x$

$= x^4 - 2x^3 + x^2 \qquad = \ln 5 + \ln x$

$\frac{du}{dx} = 4x^3 - 6x^2 + 2x \qquad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{d(uv)}{dx} = \frac{(x^2-x)^2}{x} + (\ln 5x)(4x^3 - 6x^2 + 2x)$$

$$= \frac{x^4 - 2x^3 + x^2}{x} + (\ln 5x)(4x^3 - 6x^2 + 2x)$$

$$= \boxed{x^3 - 2x^2 + x + (\ln 5x)(4x^3 - 6x^2 + 2x)}$$

Q6

Product rule $\quad \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$y = uv$

Let $u = f(x)$ $\quad v = (g(x))^{-1}$

chain rule $(u = g(x))$ $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{du}{dx} = f'(x)$ $\quad \frac{dv}{dx} = (g'(x))(-g(x)^{-2})$

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = f(x)g'(x)\left(-(g(x))^{-2}\right) + (g(x))^{-1}f'(x)$$

$\times \frac{g(x)}{g(x)}$

$$= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)g(x)}{(g(x))^2}$$

$$\boxed{\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

Q7a

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

a) $x = \frac{1}{\cos 7y} = (\cos 7y)^{-1}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

flip around the fractions
OR
exchange the $x$s and $y$s

Chain rule

$\frac{dx}{dy} = \frac{dx}{du} \times \frac{du}{dy}$

Let $x = u^{-1}$ $\quad u = \cos 7y$

$\frac{dx}{du} = -u^{-2}$ $\quad \frac{du}{dy} = -7 \sin 7y$

$\frac{dx}{dy} = -u^{-2}(-7\sin 7y) = \frac{7\sin 7y}{u^2}$

$= \frac{7\sin 7y}{(\cos 7y)^2} = 7\tan 7y \sec 7y$

$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{7\tan 7y \sec 7y}$

$$\boxed{\frac{dy}{dx} = \frac{1}{7\tan 7y \sec 7y}}$$

Q7b

,

$\frac{dy}{dx} = \frac{1}{7\tan 7y \sec 7y}$

ns of $x$.

$\sqrt{x^2+1}$ $\quad x$

[2]

b) $x = \sec 7y$

Find $\tan 7y$ in terms of $x$

$\sec^2 7y = \tan^2 7y - 1$ $\quad$ trig identity!

$x^2 = \tan^2 7y - 1$

$x^2 + 1 = \tan^2 7y$

$\sqrt{x^2+1} = \tan 7y$

$$\boxed{\frac{dy}{dx} = \frac{1}{7x\sqrt{x^2+1}}}$$

Q8

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Let $y = \frac{u}{v}$

$u = \sin x \qquad v = 1 - e^x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -e^x$

$$\frac{dy}{dx} = \frac{(1-e^x)\cos x - (\sin x)(-e^x)}{(1-e^x)^2} = 0$$

$$\frac{\cos x - e^x\cos x + e^x\sin x}{1 + e^{2x} - 2e^x} = 0$$

$$\boxed{\frac{\cos x + e^x(\sin x - \cos x)}{e^{2x} - 2e^x + 1} = 0}$$

Q9A

a) The derivative of $\tan^{-1}x$ is $\frac{1}{1+x^2}$ (this can be shown using implicit differentiation).

So applying the chain rule,

Let $u = \frac{x}{a}$

$$\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2} \times \frac{du}{dx}$$

Sub $u = \frac{x}{a}$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{1}{a + \frac{x^2}{a}}$$

$$\boxed{\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}}$$

Q9b

b) Using $a = 2$ in the result from (a).

$$\frac{dy}{dx} = -\frac{1}{4} + \frac{2}{4+x^2} = 0$$

$$\frac{2}{4+x^2} = \frac{1}{4}$$

$$8 = 4 + x^2$$

$$4 = x^2$$

$$x = \pm 2$$

When $x = 2$,

$$y = -\frac{2}{4} + \tan^{-1}\left(\frac{2}{2}\right) = -\frac{1}{2} + \frac{\pi}{4}$$

$x = -2$, $y = -\frac{(-2)}{4} + \tan^{-1}\left(\frac{-2}{2}\right) = \frac{1}{2} - \frac{\pi}{4}$

SPs: $\boxed{\left(2,\ \frac{\pi}{4} - \frac{1}{2}\right) \text{ and } \left(-2,\ -\frac{\pi}{4} + \frac{1}{2}\right)}$